Consumer Theory Application I: Uncertainty & Insurance

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We have examined cases where outcomes are *certain*. What if outcomes are not guaranteed, but are *uncertain* (i.e. probabilistic)? Things like your health, your investments, and the integrity of your property (from theft, natural disasters, war, etc) certainly fall into this category. For this reason, we take out *insurance* on valuable things that have some probability of significantly losing their value. Let's explore how Consumer Theory can make our actions more intelligible:

1 Review of Mathematical Statistics: Expected Value

The **expected value** of any uncertain outcome is the sum of the product of each possible outcome and its associated probability of ocurrence

$$E(X) = x_1p_1 + x_2p_2 + ... + x_np_n = \sum_{i=1}^{n} x_ip_i$$

We often call the situation being considered a **lottery**, implying a payout (x) is earned with some probability, just like an actual State lottery where you buy tickets and have a very very small probability of winning a lot of money.

We say a lottery is **actuarially fair** if the price of participating in the lottery (e.g. buying a lottery ticket) is equal to your expected value.

1.1 Example

Suppose you loan a friend \$100, and they promise to pay it back a month later along with \$10 interest. There is, however, a 10% probability that they can not pay and thus you get nothing. What is the expected value of your repayment in a month?

Payout x_i	Probability $p(x_i)$
\$0	0.10
\$110	0.90

$$E(x) = (\$0 * 0.10) + (\$110 * 0.90)$$
$$E(x) = (\$0) + (\$99)$$
$$E(x) = \$99$$

1.2 Example

Suppose you may earn the following amounts of money with the following corresponding probabilities:

Payout x_i	Probability $p(x_i)$
\$0	0.20
\$100	0.60
\$200	0.20

What is the expected value?

$$E(x) = (\$0 * 0.20) + (\$100 * 0.60) + (\$200 * 0.20)$$
$$E(x) = (\$0) + (\$60) + (\$40)$$
$$E(x) = \$100$$

1.3 Technical & Historical Note

A natural extension of the model of utility-maximizing consumers (in the presence of certainty) is to assume that in the presence of uncertainty, individuals **maximize expected value**.

The expected-value theory ran into some bizarre paradoxes, such as the *St. Petersburg Paradox*:

Suppose a casino offers the following game: A fair coin is tossed. If tails, the game ends, and you win \$2. If heads, the pot is doubled. You toss the coin again, and if tails, you win \$4, if heads, the pot is again doubled. And so on...What would be a fair price for the casino to charge to play the game?

$$E(x) = \frac{1}{2} \times \$2 + \frac{1}{4} \times \$4 + \frac{1}{8} \times \$8 + \frac{1}{16} \times \$16 + ...$$
$$E(x) = 1 + 1 + 1 + 1 + ...$$
$$E(x) = \infty$$

2 Expected Utility Theory

A more robust theory was discovered by John von Neumann and Oskar Morgenstern, which instead of mere expected value, rational individuals are assumed to *maximize* ***expected utility***. This imposes a utility function that describes an individuals preferences over possible expected outcomes. In other words, the expected utility function represents one's preferences over lotteries.

To calculate the expected utility function, similar to how we calculate expected value, we take the sum of the product of the person's *utility* of earning one of the possible lottery outcomes a possible and its associated probability.

$$EU(x) = u(x_1)p_1 + u(x_2)p_2 + ... + \sum_{i=1}^{n} u(x_i)p_i$$

Notice the only difference between expected utility and expected value is that expected value multiplies each payout times its probability; expected utility multiplies the person's *utility* of each payout times its probability.

This requires having a utility function $u(x)$ describing how much each payout generates in utility. Again, this is an ordinal function, that requires only that:

$$u(x_1) > u(x_2) \iff x_1 \succ x_2$$

that is, earning higher utility from payoff 1 than utility from payoff 2 means payoff 1 is preferred to payoff 2. These are sometimes called Von Neumann-Morgenstern utility functions.

We often assume that individuals are **risk-averse**, that is, that they prefer a certain outcome over an uncertain outcome (sometimes even if the payoff from the uncertain outcome is higher than the certain one!). Not all individuals are risk averse, we can also consider someone who is risk loving, or risk neutral.

2.1 Example

Suppose utility (U) is a function of income (I):

$$u(I) = \sqrt{I}$$

Suppose income is \$100 per day, but there is a 25% chance of illness or injury, which would leave a net income of \$4.

Expected income (simple expected value) is:

$$E(I) = (0.75)100 + (0.25)\$400$$
$$E(I) = (\$75) + (\$10)$$
$$E(I) = \$76$$

Now if we take the expected utility, it would be the sum of the utility of each payout times its probability:

$$u(I) = (0.75)u(\$100) + (0.25)u(\$4)$$
$$u(I) = (0.75)\sqrt{\$100} + (0.25)\sqrt{\$4}$$
$$u(I) = (0.75)(10) + (0.25)(2)$$
$$u(I) = 7.5 + 0.5$$
$$u(I) = 8$$

Compare, however, the utility of the expected income, i.e. take the utility of the expected value of the lottery–\$76. The utility of having the \$76 *if it were certain* would be:

$$u(I) = \sqrt{I}$$
$$u(\$76) = \sqrt{76}$$
$$u(\$76) \approx 8.72$$

Note the utility of having the *certain* \$76 is higher than utility of the *uncertain* outcome. This implies that the person is **risk averse**: they would prefer income that is *certain* to income that is *uncertain*, even if the certain income may be lower than the uncertain income.

If we look at this graphically, this implies that a risk-averse utility function is **concave**. A line connecting any two points on the function will always lie *under* the function–the mathematical definition of concave.

The two possible outcomes are plotted as points on Figure 1 below. Point A represents the \$100 income. The utility of having \$100 is:

$$u(I) = \sqrt{I}$$
$$u(\$100) = \sqrt{100}$$
$$u(\$100) = 10$$

Point B represents the \$4 income.

$$u(I) = \sqrt{I}$$
$$u(\$4) = \sqrt{4}$$
$$u(\$4) = 2$$

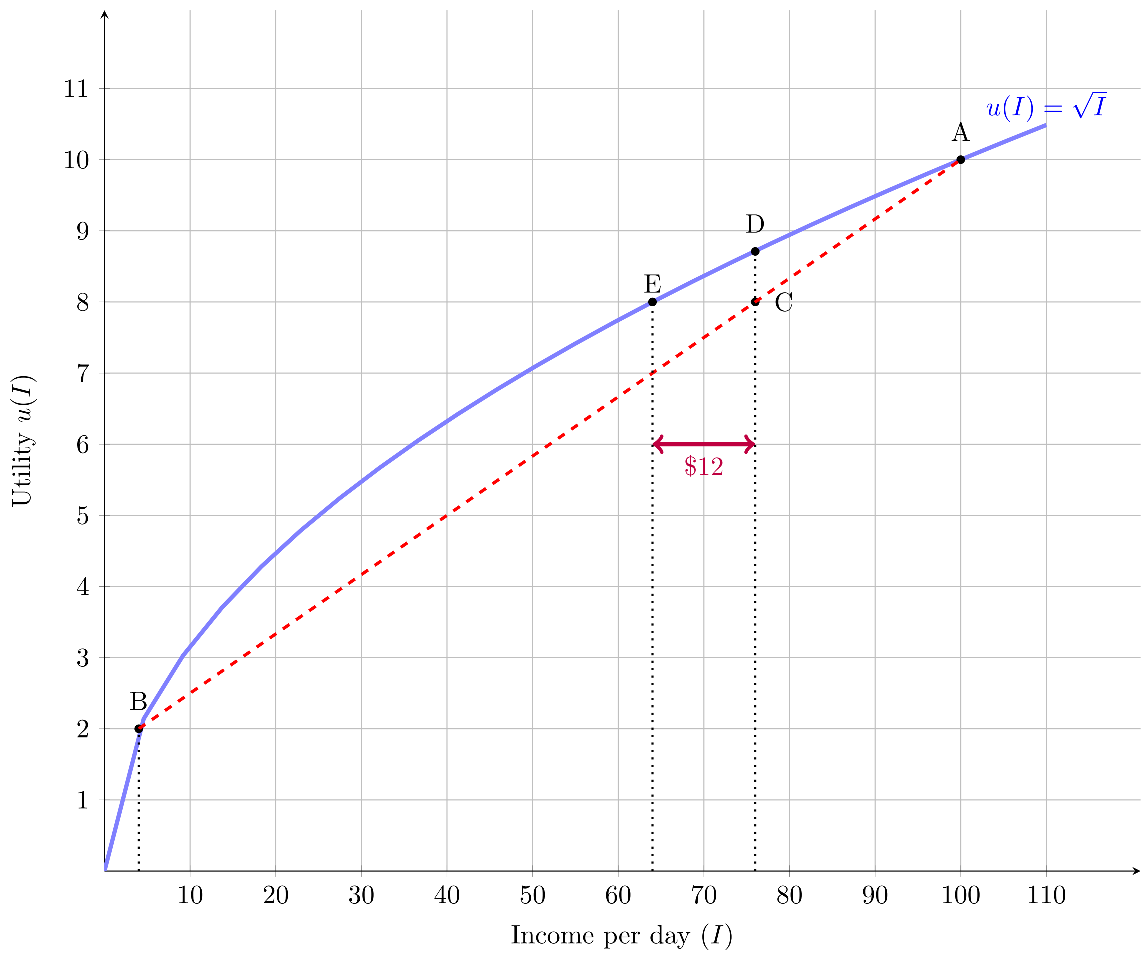

Figure 1: Utility function for risk-averse individual

If we connect these two points with a (red dashed) line, we can take a weighted average of these two states of the world based on their probability of occurrence (75% chance of A, 25% chance of B). This weighted average (the expected value) is represented with point C at \$76. We saw that the overall expected utility of this lottery is 8.

Note, however, we also looked at the utility of having \$76 (the expected value) with *certainty*, which was 8.72. This is plotted as point D.

Of course, the \$76 is *not* guaranteed. We can ask what the **certainty equivalent** of having the uncertain \$76, that is, what amount of *guaranteed* income would make you indifferent to (e.g. have the same amount of utility as) having the uncertain \$76?

$$\begin{aligned} u(I) &= \sqrt{I} \\ 8 &= \sqrt{I} \\ 64 &= I \end{aligned}$$

You would need a guaranteed \$64 amount of income to get the same utility as your expected (but uncertain) \$76. Thus, \$64 is the **certainty equivalent** of this lottery. This is plotted as point E on the graph, note it has the same utility as C.

2.2 Insurance

Note that the difference between point E (the certainty equivalent) and point C (the expected value) is \$12. We can think of this as the **actuarially fair premium** for **insurance**. That is, this person is willing to make payments of \$12 (the "premium") to an insurance company that will compensate the person if the person experiences loss of income (the 25% probable event of \$4).

A risk averse person would choose to pay insurance premiums to guarantee a certain level of income (\$64). In this case, the risk (of this person suffering an adverse event and only earning \$4) is borne by the insurance company, rather than the individual.

2.3 Other Attitudes Towards Risk

We have simply assumed that the individual is risk averse (the average person probably is, but not everyone). We can express a person as being **risk loving** or **risk neutral**. We have seen that a risk averse person has a concave utility function – the expected value of a risky outcome (point C) provides lower utility than having that same amount of income with certainty (point D); mathematically, a line connecting two points on the function always lies *below it*.

A risk loving person would have a utility function bending in the opposite direction: they would get *higher* utility from the expected value of a risky outcome than from that same amount of income with certainty. This makes the function *convex*: a line connecting two points on the function lies *above* the function. A risk loving person would *not* buy insurance, because they *prefer* a risky outcome than an equivalent certain outcome.

A risk neutral person would get the *same* utility from the expected value of a risky outcome as an identical certain outcome. This makes the function *linear*. A risk neutral person would *not* buy insurance, because they are indifferent between a risky and an equivalent certain outcome.

All three cases can be seen in Figure 2 below.

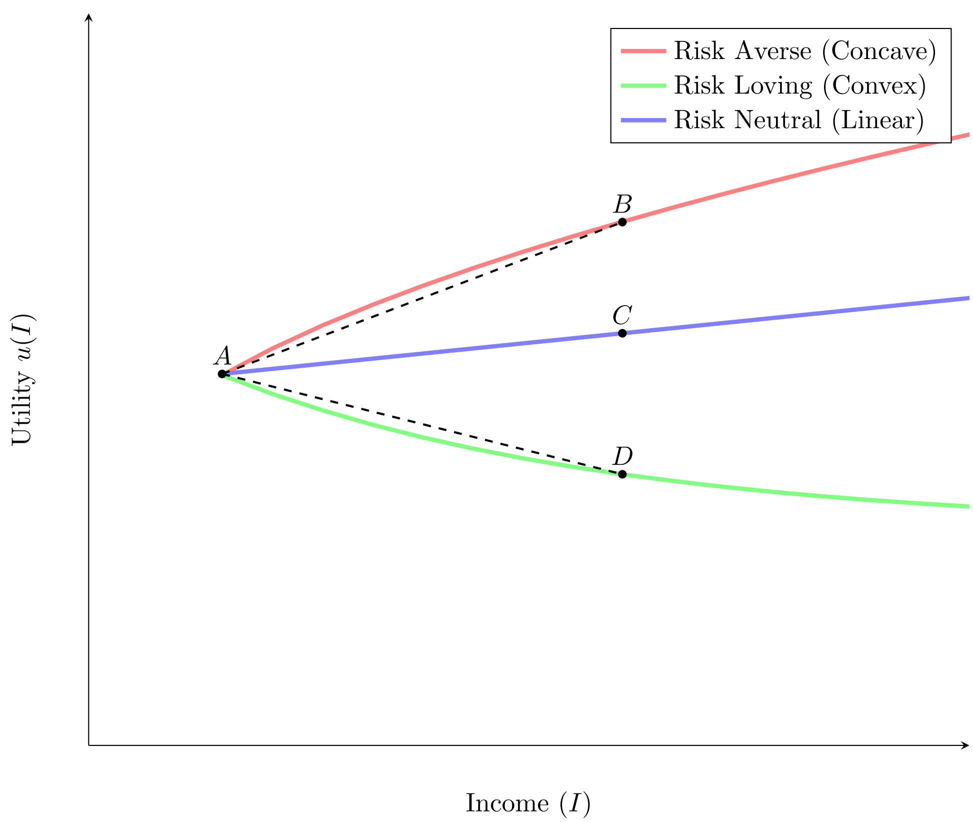

Figure 2: Utility functions for different risk attitudes